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ON THE CONVERGENCE OF THE CONTINUED FRACTION OF GAUSS AND OTHER CONTINUED FRACTIONS.*

BY EDWARD B. VAN VLECK.

IN the *Disquisitiones generales circa seriem infinitam* . . . of Gauss the quotient

$$G(\alpha, \beta, \gamma, x) = \frac{F(\alpha, \beta + 1, \gamma + 1, x)}{F(\alpha, \beta, \gamma, x)}$$

was developed into the continued fraction

$$\frac{1}{1\, -}\, \frac{c_2 x}{1\, -}\, \frac{c_3 x}{1\, -} \cdots,$$

in which

$$c_{2n} = \frac{(\alpha + n - 1)(\gamma - \beta + n - 1)}{(\gamma + 2n - 2)(\gamma + 2n - 1)}, \quad c_{2n+1} = \frac{(\beta + n)(\gamma - \alpha + n)}{(\gamma + 2n - 1)(\gamma + 2n)}.$$

The convergence of this continued fraction was later a subject of investigation by Thomé[†] and Riemann.[‡] Both writers showed that, if the portion of the real axis between $x = 1$ and $x = +\infty$ is regarded as a cut, the continued fraction will converge in the remainder of the plane of x except at the points in which $F(\alpha, \beta, \gamma, x)$ and its analytic continuation vanish, and that the limit of the continued fraction is the analytic function defined by $G(\alpha, \beta, \gamma, x)$.[§] The proofs of Thomé and Riemann are both of a special and somewhat involved character, not based upon considerations which can be applied to prove the convergence of any considerable class of continued fractions. Indeed, it is only within the past decennium that general theorems for the convergence

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[*] This paper was read before the American Mathematical Society at its summer meeting in Ithaca, Aug. 20, 1901.

[†] *Crelle*, vol. 67 (1867), p. 299.

[‡] Sullo svolgimento del quoziente di due serie ipergeometriche in frazione continua infinita, *Gesammelte Werke*, 1st ed., p. 400. The completion of this posthumous fragment is due to Schwarz.

[§] An elementary proof for the special case in which $0 < x < 1$ is found in Schlömilch, *Algebr. Analysis*, p. 321.

of algebraic* continued fractions† have been obtained. The object of this paper is to demonstrate the convergence of the continued fraction of Gauss by theorems which are of the latter character. The theorems employed for this purpose are new and cover a wide class of algebraic continued fractions. Further application of these theorems is made to the quotient of two related Bessel's functions, J_{n-1}/J_n, and to the continued fraction which Heine has given as a generalization of that of Gauss.

I. The Fundamental Theorems.

1. *Statement of the First Two Theorems.* To most of the common algebraic continued fractions one or both of the two following theorems may be applied:

Theorem 1. *If k denotes the greatest modulus of a point of condensation of the coefficients a_n of the continued fraction*

$$\frac{a_1}{1+}\,\frac{a_2x}{1+}\,\frac{a_3x}{1+}\cdots\cdot, \tag{1}$$

the continued fraction will represent an analytic function within a circle of radius $1/4k$, described about the origin of the x-plane as centre, and the only singularities of the function contained within the circle will be poles.

If, furthermore, any circle of smaller radius is drawn about the origin and from this circle each of the poles is excluded by drawing around it a small but arbitrary contour, then within the region remaining the continued fraction will converge uniformly to the analytic function as its limit.

Theorem 2. *If from and after some fixed point in the continued fraction*

$$\frac{1}{b_1z+}\,\frac{1}{b_2+}\,\frac{1}{b_3z+}\,\frac{1}{b_4+}\cdots\cdot\cdot \tag{2}$$

b_n is real and positive,‡ and $\sum_{n=1}^{\infty} b_n$ is divergent, the continued fraction will converge over the entire plane of z with the exception of

(1) *the whole or a part of the negative half of the real axis;*

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* Continued fractions whose partial quotients are functions of one or more variables.

† A summary of the few theorems obtained hitherto will be found in the *Transactions of the American Mathematical Society*, vol. 2, 1901, p. 215.

‡ The theorem holds also if from and after some fixed point b_n is negative.

(2) *a series of isolated points* $p_1, p_2, \ldots$, *which lie without this half of the axis.**

The convergence, moreover, is uniform in any region from which these points and the negative half-axis are excluded after the manner of Theorem 1, *and the limit of the continued fraction is an analytic function whose only singularities exterior to the negative half-axis are the points* $p_1, p_2, \ldots$, *which are poles.*

2. *Proof of these Theorems.* The proof of Theorem 1 will be found in the *Transactions of the American Mathematical Society*, vol. 2, October, 1901. Theorem 2 may be demonstrated as follows.

By hypothesis, there is some point in (2) subsequent to which b_n is real and positive. Suppose this to be true when $n \geqq 2\mu$, and consider the fraction

$$\frac{1}{b_{2\mu+1}z} + \frac{1}{b_{2\mu+2}} + \frac{1}{b_{2\mu+3}z} + \cdots \qquad (3)$$

If the nth convergent of (2) is denoted by N_n/D_n and the mth convergent of (3) by N'_m/D'_m, we obviously have

$$\frac{N_{2\mu+m}}{D_{2\mu+m}} = \frac{N_{2\mu} + \dfrac{N'_m}{D'_m} N_{2\mu-1}}{D_{2\mu} + \dfrac{N'_m}{D'_m} D_{2\mu-1}}. \qquad (4)$$

Hence to ascertain whether the successive convergents of (2) approach a limit, it is sufficient first to determine whether this is the case with the convergents of (3).

Now Stieltjes† has proved that when the coefficients $b_{2\mu+n}$ in the continued fraction (3) are positive for all values of n and $\Sigma\, b_{2\mu+n}$ is divergent, the continued fraction will converge uniformly in any region T, bounded by a closed curve which has no point in common with the negative half of the axis of z, and the limit of the fraction is a function which is holomorphic in T. It follows that when m is indefinitely increased, N'_m/D'_m converges uniformly within any such region, and its limit $\phi(z)$ is holomorphic at every point without the negative half-axis. Since the numerator and denominator of the right hand member of (4) also converge in the same manner, their quotient must converge uniformly to the limit

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* It is proved later (§3) that if b_n fulfills the conditions of the theorem and is also real from the beginning of the continued fraction, the number of the points p_n must be finite. Whether this number is always finite does not appear; but if the number is infinite, the derivative of the point-set obviously can consist only of points which lie upon the negative half of the axis.

† *Annales de Toulouse*, vol. 8. Another proof is given by the writer in the *Transactions Amer. Math. Society*, vol. 2, 1901; see in particular p. 231–2.

$$\frac{N_{2\mu} + \phi(z)\ N_{2\mu-1}}{D_{2\mu} + \phi(z)\ D_{2\mu-1}} \tag{5}$$

except in the vicinity of the roots of $D_{2\mu} + \phi(z)\ D_{2\mu-1}$ or along the negative half of the axis. It is impossible for $D_{2\mu} + \phi(z)\ D_{2\mu-1}$ to vanish identically, since in that case the numerator would have to vanish identically too, and hence $N_{2\mu} D_{2\mu-1} - N_{2\mu-1} D_{2\mu} \equiv 0$, which is not true. As the numerator and denominator of (5) are also holomorphic at all points without the half-axis, the expression (5) is an analytic function whose only singularities exterior to the half-axis are poles. The continued fraction (2) therefore converges in the manner stated in the theorem.*

3. *A Sufficient Condition that the Number of the Poles be Finite.* In at least one case (which is stated below in Theorem 3) we can easily prove that the number of poles $p_1, p_2, \ldots$ is finite.

When, namely, the coefficients in the continued fraction (2) are real for all values of n and satisfy the conditions of Theorem 2, an upper limit can be found for the number of poles of the limiting function exterior to the negative half of the axis. These poles, in fact, correspond to the zeros of the analytic function to which D_n converges as n increases indefinitely, and the convergence is obviously uniform in the vicinity of each zero which does not lie upon the negative half-axis. Now it is well known that when a series of functions $f_n(z)$, holomorphic throughout any region T, converges uniformly in this region, the zeros of the limiting function within T are the points of condensation of the zeros of the functions $f_n(z)$. Here we have $f_n(z) = D_{2\mu} + \frac{N'_n}{D'_n} D_{2\mu-1}$. Furthermore, Hurwitz† has shown that if a circle of sufficiently small radius is drawn about each zero of the limiting function within T, each of the circles will contain a number of roots of $f_n(z)$ exactly equal to the order of the zero enclosed, for all values of $n \geqq m$, where m is a fixed integer. Our question concerning the number of poles is thus reduced to a determination of an upper limit for the number of roots of D_n exterior to the negative half of the axis.

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* For the sake of completeness it may be added that when $\Sigma\, b_n$ is convergent, the even convergents of (2) and the odd convergents approach separate limits, so that it is impossible to speak of the convergence of the continued fraction or of a single limiting function which it represents. This is always the case when Σb_n converges, irrespective of other conditions which may be imposed on b_n, and the limits of the two series of convergents are meromorphic over the entire plane. This result is not new. See, for example, the memoir of Stieltjes previously cited. For another mode of proof see the *Transactions Amer. Math. Soc.*, vol. 2, 1901, p. 231.

† *Math. Annalen*, vol. 33, 1888, p. 249.

Let ν be the smallest value of n subsequent to which b_n is always positive, and consider then the number of changes of sign in the series

$$D_\nu, \quad D_{\nu+2}, \quad D_{\nu+4}, \; \ldots, \; D_{\nu+2n} \tag{6}$$

for any real value of z. When z varies continuously, this number will change only when one or more of the D's vanish. From the fundamental relation connecting the denominators of three successive convergents it will be found that

$$D_n = \left(b_n b_{n-1} z + 1 + \frac{b_n}{b_{n-2}} \right) D_{n-2} - \frac{b_n}{b_{n-2}} D_{n-4}, \tag{7}$$

alike when n is odd or even. Accordingly, when any term in (6) vanishes, the preceding and following terms have opposite signs. If two successive terms should vanish, then simultaneously every $D_{\nu \pm 2i}$ vanishes. But $D_0 \equiv 1$, $D_1 \equiv b_{2\mu+1} z$. This happens therefore only when ν is odd and $z = 0$. Hence we see that an alteration in the number of changes of sign in (6) can take place only when z passes through a root of D_ν or $D_{\nu+2n}$, and that there is a loss or a gain of a single sign for a root of either unless $z = 0$.

Suppose now that ν is even. If we throw out a common factor, the terms of highest degree in the successive members of (6) will be equal to

$$1, \quad b_{\nu+1} b_{\nu+2} z, \quad b_{\nu+1} b_{\nu+2} b_{\nu+3} b_{\nu+4} z^2, \; \ldots\ldots, \; b_{\nu+1} \ldots b_{\nu+2n} z^n. \tag{8}$$

When, therefore, $z = -\infty$ is substituted in (6), there will be n alterations of sign. On the other hand, the terms of (6) all have the same sign when $z = 0$, since the constant term of D_{2n} is always equal to 1. It follows that D_ν and $D_{\nu+2n}$ together must have at least n roots in the negative half-axis. But their degrees are equal respectively to $\nu/2$ and $n + \nu/2$. We conclude therefore that the number of roots of $D_{\nu+2n}$ without the half-axis can not exceed ν.

Suppose next that ν is odd. As before, there will be n changes of sign in (6) when $z = -\infty$. The terms of lowest degree, however, are equal now to

$$(b_1 + b_3 + b_5 + \cdot\cdot + b_\nu)z, \quad (b_1 + b_3 + \cdot\cdot + b_{\nu+2})z, \cdot\cdot\cdot, \quad (b_1 + b_3 + \cdot\cdot + b_{\nu+2n})z.$$

Since b_n, by hypothesis, is positive for $n > \nu$, the number of changes of sign here can not exceed unity. Hence the number of changes of sign in (6) for an exceedingly small negative value of z cannot exceed unity, and D_ν and $D_{\nu+2n}$ together have at least $n - 1$ roots between $z = 0$ and $z = -\infty$. As they also each have a root in the origin and as their degrees are equal respectively to $(\nu + 1)/2$ and $n + (\nu + 1)/2$, we conclude again that the number of roots of D_n exterior to the negative half of the axis will not exceed ν. We reach thus the following result:

THEOREM 3 *If the coefficients b_n are real from the outset in the continued fraction of Theorem 2 and if ν is the smallest value of n subsequent to which they are always positive, the number of poles of the function defined by the continued fraction, exterior to the negative half of the real axis of z, will not exceed ν, each multiple pole being counted a number of times equal to its order.*

4. *A Sufficient Condition for the Applicability of Theorem* 1. We proceed now to determine a class of continued fractions of the form (1) to which Theorem 1 is applicable. In many continued fractions of that form, it is possible, for sufficiently large values of n, to expand a_{2n-1}/a_{2n} and a_{2n}/a_{2n+1} into convergent series in ascending powers of $1/n$ with constant coefficients. Suppose then that

$$\frac{a_{2n-1}}{a_{2n}} = 1 + \frac{A_1'}{n} + \frac{A_2'}{n^2} + \cdot\cdot\cdot\cdot, \tag{9}$$

$$\frac{a_{2n}}{a_{2n+1}} = 1 + \frac{A_1''}{n} + \frac{A_2''}{n^2} + \cdot\cdot\cdot\cdot \tag{10}$$

From these we derive

$$\frac{a_{2n+1}}{a_{2n-1}} = 1 - \frac{A_1' + A_1''}{n} + \frac{1}{n^2}(\ ) + \cdot\cdot\cdot\cdot, \tag{11}$$

and a similar power series for a_{2n+2}/a_{2n}, in which the first two terms are the same as in (11). If now the real part of $A_1' + A_1''$ is positive, a theorem of Weierstrass* shows that a_{2n+1} and a_{2n} with increasing n approach the limit 0.

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* Ueber die Theorie der analytischen Facultäten, *Crelle*, vol. 51, 1855, p. 25; *Werke*, vol. 1, p. 181.

Theorem 1 then gives complete information concerning the continued fraction and shows that it represents a function which is either holomorphic or meromorphic over the entire finite plane.

If the real part of $A_1' + A_1''$ is negative, a_n approaches the limit $k = \infty$, and no information at all concerning the continued fraction can be obtained from Theorem 1.

Lastly, if the real part of $A_1' + A_1''$ is equal to zero, the moduli of a_{2n} and a_{2n+1} approach finite limits, which by (10) are seen to be identical. Their arguments, however, will fail to approach definite values, unless the imaginary part of $A_1' + A_1''$ vanishes, too. When this happens, a_{2n} and a_{2n+1} must both approach limiting values, since $\prod_{n=1}^{\infty}(1 + \frac{1}{n^2})$ is convergent. The identity of the two limits then follows from (10). *If, therefore, equations* (9) *and* (10) *hold for large values of n, the necessary and sufficient condition that a_n shall approach a finite limit different from zero is that $A_1' + A_1''$ shall be equal to zero.*

5. *The Convergence of the Continued Fraction* (1) *when $A_1' + A_1'' = 0$. Case* I: *a_n Ultimately Real.* When the condition:

$$A_1' + A_1'' = 0$$

is fulfilled, Theorem 1 determines the convergence of the continued fraction for a portion of the x-plane.* Further information regarding its convergence can be gained by throwing it into the form (2). For this purpose put $x = 1/z$ and then transform so as to remove z from the partial numerators. It will then appear in the alternate denominators, beginning with the first. An easy and well known reduction, which alters neither the value of the continued fraction nor the value of its convergents, then gives the following values for the coefficients of (2):

$$b_1 = \frac{1}{a_1}, \quad b_{2n} = \frac{a_1 a_3 \ldots a_{2n-1}}{a_2 a_4 \ldots a_{2n}}, \quad b_{2n+1} = \frac{a_2 a_4 \ldots a_{2n}}{a_1 a_3 \ldots a_{2n+1}}. \qquad (12)$$

We will take up first the case in which a_n *is real for all values of* $n \geqq m$, where m denotes a fixed number. Since by hypothesis

$$\lim_{n=\infty} \frac{a_{n-1}}{a_n} = 1,$$

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* Theorem 1 is also applicable when only the real part $A_1' + A_1''$ vanishes. This case will not concern us here.

a_n must ultimately have a constant sign. If this sign is $-$, it may be changed to $+$ by putting $x = -x'$. We may therefore assume that a_n is real and positive when n exceeds the even integer 2μ.

Let now the first 2μ partial quotients of (1) be omitted, and consider the continued fraction which remains. For convenience we will denote the coefficients of its numerators by $a'_1, a'_2, \cdots$. The corresponding values of b'_{2n} and b'_{2n+1} are real and positive. Furthermore, when n is sufficiently increased, we have

$$\frac{b'_{2n}}{b'_{2n-2}} = \frac{a'_{2n-1}}{a'_{2n}} = 1 + \frac{A'_1}{n+\mu} + \frac{A'_2}{(n+\mu)^2} + \cdots, \tag{13}$$

$$\frac{b'_{2n+1}}{b'_{2n-1}} = \frac{a'_{2n}}{a'_{2n+1}} = 1 + \frac{A''_1}{n+\mu} + \cdots. \tag{14}$$

Now since $A'_1 + A''_1 = 0$, either A'_1 or A''_1 must be greater than -1; say the former. Then

$$\lim_{n=\infty} n\left(\frac{b'_{2n}}{b'_{2n-2}} - 1\right) > -1,$$

whence it follows by a theorem of Raabe's* that $\sum\limits_{n=1}^{\infty} b'_{2n}$ is divergent. If $A''_1 > -1$, we see similarly that $\Sigma\, b'_{2n+1}$ diverges. In either case, all the conditions of the theorem of Stieltjes quoted in §2 are fulfilled, and the continued fraction, after the omission of the first 2μ partial quotients, will converge over the entire plane of z except along a cut from $z = 0$ to $z = -\infty$. Since $x = 1/z$, there is a similar region of convergence in the plane of x. When the 2μ partial quotients are restored, the reasoning of §2 may be repeated. We conclude therefore that under the conditions indicated the continued fraction (1) represents a function whose only singularities exterior to the negative half-axis of x are poles.†

6. *Continuation; Case* II: a_n *Not Ultimately Real.* The direct consideration of (1) when $A'_1 + A''_1 = 0$, appears to be difficult if a_n is not ultimately real, or at least if A'_1 and A''_1 are not real. For Weierstrass* shows that the argument of b_n will fail to approach a limiting value if the imaginary component of A'_1 in (13) or of A''_1 in (14) is not equal to zero. On this account it is better to proceed indirectly.

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* *Zeitschrift für Physik und Mathematik*, vol. 10, Vienna, 1832, p. 63; *cf.* for example, Stolz, *Allgemeine Arithmetik*, vol. 1, p. 263.

† If $A'_1 < -1$ and $A''_1 < -1$, the series $\Sigma b'_n$ is convergent. In this case the limit of the series of odd convergents or of the series of even convergents is meromorphic over the entire plane of x, but the two limits are not identical.

Equation (7) enables us to consider the even and the odd convergents separately. Since a similar relation connects the numerators of the convergents of (2), the even convergents form by themselves the successive convergents of the continued fraction

$$\frac{b_2}{b_1b_2z+1-}\;\frac{\frac{b_4}{b_2}}{b_4b_3z+1+\frac{b_4}{b_2}-}\;\frac{\frac{b_6}{b_4}}{b_6b_5z+1+\frac{b_6}{b_4}-}\;\cdots, \qquad (15)$$

while the odd convergents have a like connection with

$$\frac{1}{b_1z}\left(1-\frac{\frac{b_3}{b_1}}{b_3b_2z+1+\frac{b_3}{b_1}-}\;\frac{\frac{b_5}{b_3}}{b_5b_4z+1+\frac{b_5}{b_3}}\;\cdots\right). \qquad (16)$$

If in (16) we set $\frac{b_{2n+1}}{b_{2n-1}}=a''_n$ and then reduce the continued fraction to the normal form in which each partial numerator is equal to 1, it becomes

$$\frac{1}{b_1z}\left(1-\frac{1}{b''_1\left(b_3b_2z+1+\frac{b_3}{b_1}\right)-}\;\frac{1}{b''_2\left(b_5b_4z+1+\frac{b_5}{b_3}\right)-}\;\cdots\right), \qquad (17)$$

in which b''_n is obtained from the a''_n by doubly accenting the letters of (12).

The continued fraction (17) is of the form

$$1-\frac{1}{B_1z+C_1-}\;\frac{1}{B_2z+C_2-}\;\cdots\cdots \qquad (17a)$$

We shall show that B_n approaches a limit different from 0, which we shall denote by k^{-1}, and that C_n approaches 2 as its limit, when $n=\infty$. The convergence of the continued fraction

$$1-\frac{1}{k^{-1}z+2-}\;\frac{1}{k^{-1}z+2-}\;\cdots\cdots \qquad (17b)$$

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* *Werke, l. c.*, p. 181–184.

can be dealt with directly by means of theorems already established, and then the fraction (17a) can be treated in a similar manner.

Let the first $2r - 1$ partial quotients of (1) be omitted and consider in its place the continued fraction which remains. If r has been taken sufficiently large we shall have in the continued fraction corresponding to (17)

$$b_1'' = \frac{b_1}{b_3} = \frac{a_{2r+2}}{a_{2r+1}} = 1 - \frac{A_1''}{r+1} + \frac{(\;\;)}{(r+1)^2} + \cdot \cdot ,$$

and the value of b_1'' can accordingly be made to differ from unity by as little as we choose. The same is obviously true of b_2''. Furthermore

$$\frac{b_{2n}''}{b_{2n-2}''} = \frac{a_{2n-1}''}{a_{2n}''} = \frac{b_{4n-1}}{b_{4n-3}} \div \frac{b_{4n+1}}{b_{4n-1}} = \frac{a_{4n+2r-3}}{a_{4n+2r-2}} \div \frac{a_{4n+2r-1}}{a_{4n+2r}},$$

whence we obtain with the aid of (9) and (10)

$$\frac{b_{2n}''}{b_{2n-2}''} = \frac{1 + \dfrac{A_1'}{2n+r-1} + \dfrac{A_2'}{(2n+r-1)^2} + \cdot \cdot \cdot}{1 + \dfrac{A_1'}{2n+r} + \dfrac{A_2'}{(2n+r)^2} + \cdot \cdot \cdot}$$

$$= 1 + \frac{B_2}{(2n+r)^2} + \frac{B_3}{(2n+r)^3} + \cdot \cdot \cdot \cdot$$

It will be noticed that the first power of $1/(2n + r)$ does not appear. We can show in like manner that it is absent in the expansion of b_{2n+1}''/b_{2n-1}''. There must therefore exist a positive number C such that the inequality

$$\left| \frac{b_n''}{b_{n-2}''} \right| < 1 + \frac{C}{(n+r)^2}$$

shall hold for all values of n. Now the product

$$\prod_{n=1}^{\infty} \left(1 + \frac{C}{(n+r)^2}\right)$$

by increasing r sufficiently can be made to differ from unity by as little as we please. Since this is also the case with b_1'' and b_2'', it is possible so to choose r as to make

$$|b_n'' - 1| < \epsilon_1, \qquad (n = 1, 2, \ldots),$$

in which ϵ_1 is a small but arbitrarily assigned positive quantity.

Furthermore, in each partial denominator of (17) we have the relation $b_n b_{n+1} = 1/a_{n+2r}$. If, then, k is the limit of a_n, the following condition may also be fulfilled

$$|b_n b_{n+1} - k^{-1}| < \epsilon_1, \qquad (n = 1, 2, \ldots)$$

Lastly, since

$$\frac{b_{n+2}}{b_n} = \frac{a_{n+2r}}{a_{n+2r+1}},$$

the value of this ratio can be made to differ from unity by a quantity of as small a modulus as is desired. Hence we see, finally, that *if r has been taken sufficiently large, the successive partial denominators of* (17) *for any given finite value of z will all differ from $k^{-1}z + 2$ by a quantity whose modulus is smaller than an arbitrarily prescribed quantity ϵ. It is also possible so to choose r that this shall hold uniformly for a given finite region of the plane of z.*

7. *Completion of the Proof in Case* II. We now have all of the material at hand for the final determination of the region of convergence of the continued fraction (1). We suppose that r fulfills the condition of the close of §6 and place

$$w = k^{-1}z.$$

Also let T' be the corresponding region in the plane of w. This region shall lie wholly above or else wholly below the real axis. If the continued fraction (17) is transformed so as to make each of its numerators equal to $+1$, the signs of the alternate denominators must be changed. Hence at each point of T' the imaginary component of the denominators will be alternately positive and negative. Also, in T' the absolute value of the ratio of the imaginary to the real component of any denominator will never sink below a finite minimum. Lastly, the sum of the absolute values of the denominators is divergent throughout T'. When these three conditions are fulfilled, it follows by a theorem which I have given in the *Transactions of the American Mathematical Society*, vol. 2, p. 229, that the continued fraction converges throughout T'. The reasoning of the succeeding pages also shows that the convergence is uniform throughout T'.

We have thus shown, under the conditions stated, that if a sufficient number of partial quotients at the outset in (1) are omitted, the convergents of odd order will converge at each point of T'. A similar course of reasoning applied to (15) will prove that this is also true of the even convergents. Since the convergence is uniform, the limits are holomorphic within T'. If, now, the omitted partial quotients are restored, the reasoning of §2 makes it evident that the limits of the two series of convergents are analytic functions whose only singularities in the interior of T' are poles. Since $w = k^{-1}x^{-1}$, it follows from Theorem 1 that the limits of the two series of convergents are identical in the portion of the w-plane which lies without a circle described about the origin as center with a radius equal to 4. If T' does not extend into this portion of the plane, it is possible to enlarge it so that it shall. Thus the limits of the odd and of the even convergents must coincide throughout the whole region T' since they are analytic functions and coincide throughout a part of this region. Finally T' can be enlarged so as to include as much of the positive (or of the negative) half-plane as is desired. We conclude therefore that if a cut is made along the real axis from $w = -4$ to $w = 4$, the continued fraction will represent a function which is meromorphic in the remainder of the plane.

To obtain the corresponding region of the x-plane, we first draw a straight line through the points $x = \pm 1/4k$ and then cut the plane along the two segments of the line which lie between these points and ∞.

8. Our last step will be to show that one of the two cuts just made may be dispensed with. Take for this purpose any region T'' in the w-plane in which the real part of w is positive and which includes a portion of the real axis. We have already proved that by properly choosing r the denominators in the continued fraction (17) can be made to differ from $w + 2$ throughout T' by a quantity whose modulus is smaller than an arbitrarily prescribed quantity ϵ. If ϵ is taken sufficiently small, the moduli of the denominators will exceed 2 at each point of T'', inasmuch as the real part of w is positive. This will also hold for each denominator, except the first, in the continued fraction which we derive from (15) in the same manner as (17) was derived from (16). The first denominator, $b_2^{-1}(b_1 b_2 k w + 1)$, will obviously exceed 1 by at least some small quantity h, if r is sufficiently large. The convergence of both continued fractions within T'' then follows from a theorem of Pringsheim.* We shall, however, reproduce his reasoning here in such form as to make it immediately apparent that the convergence of both fractions is uniform.

For convenience, let the nth partial quotient in either continued fraction be denoted by $1/r_n e^{i\theta_n}$. According to our hypothesis we have for each point of T''

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* *Sitzungsberichte der Münchener Akademie*, vol. 28, 1898, p. 313–316.

$$r_1 > 1 + h,$$

$$r_n \geqq 2 \qquad (n > 1),$$

and hence

$$|D_2| - |D_1| \geqq h.$$

From the relation

$$D_n = r_n e^{i\theta_n} D_{n-1} + D_{n-2}$$

it follows that

$$|D_n| - |D_{n-1}| \geqq |D_{n-1}| - |D_{n-2}| \geqq \cdots \geqq h.$$

Hence $|D_n| \geqq 1 + nh$. Furthermore

$$\frac{N_{n+1}}{D_{n+1}} - \frac{N_n}{D_n} = -\frac{1}{D_n D_{n+1}}$$

and consequently

$$\left|\frac{N_{n+i}}{D_{n+i}} - \frac{N_n}{D_n}\right| \leqq \frac{1}{|D_n D_{n+1}|} + \frac{1}{|D_{n+1} D_{n+2}|} + \cdots + \frac{1}{|D_{n+i-1} D_{n+i}|}$$

$$\leqq \frac{1}{h^2}\left[\frac{1}{(n+h^{-1})(n+1+h^{-1})} + \frac{1}{(n+1+h^{-1})(n+2+h^{-1})} + \cdots + \frac{1}{(n+i-1+h^{-1})(n+i+h^{-1})}\right].$$

Since the value of the last expression can be made as small as we choose by sufficiently increasing n, the continued fraction will converge uniformly within T''.

The region T'' can now be enlarged so as to include as much of the positive half of the w-axis as is desired. The portion of the w-cut between the origin and $w = 4$ was therefore superfluous. In the x-plane we may omit the cut between $x = 1/4k$ and $x = \infty$ which corresponded to this portion.

This completes our determination of the character of the function represented by (1) when $A_1' + A_1'' = 0$. The result reached can be summed up as follows:

THEOREM 4. *If in the continued fraction of Theorem* 1 *the ratios* a_{2n-1}/a_{2n} *and* a_{2n}/a_{2n+1} *for sufficiently large values of n can be expanded into convergent series in powers of* $1/n$ *and if* $A_1' + A_1'' = 0$, *the region within which the continued fraction represents an analytic function can be extended so as to include the entire plane of x with the exception of a cut drawn from* $x = -1/4k$ *to* $x = \infty$

in a direction which forms the continuation of the line joining the origin to $x = -1/4k$.

The only singularities of the analytic function exterior to the cut are poles and the continued fraction converges in the same manner as in Theorem 1.

If $A_1' + A_1'' \neq 0$ *and the real part of* $A_1' + A_1''$ *is positive, the continued fraction represents an analytic function which, in the finite region of the plane, has no other singularities than poles.*

If the real part of $A_1' + A_1''$ *vanishes but the imaginary part does not, then the continued fraction represents an analytic function which at least within a circle described with a radius* $1/4k$ *about the point* $x = 0$ *has no other singularities than poles.*

II. Application of the Fundamental Theorems.

9. (*a*). *The Continued Fraction of Gauss.* The last theorem can be applied directly to the continued fraction of Gauss. For since

$$a_{2n} = -\frac{(a-1+n)(\gamma-1-\beta+n)}{(\gamma+2n-2)(\gamma+2n-1)}, \qquad a_{2n+1} = -\frac{(\beta+n)(\gamma-a+n)}{(\gamma+2n-1)(\gamma+2n)},$$

a_n has the limit $k = -1/4$. Moreover,

$$\frac{a_{2n-1}}{a_{2n}} = 1 + \frac{2\beta - 2a + 1}{n} + \frac{1}{n^2}(\quad) + \cdots,$$

$$\frac{a_{2n}}{a_{2n+1}} = 1 - \frac{2\beta - 2a + 1}{n} + \cdots,$$

and $A_1' + A_1'' = 0$. The continued fraction therefore represents a function whose only singularities exterior to the portion of the real axis between $x = 1$ and $x = +\infty$ are poles. It also converges uniformly to the function as its limit in any region bounded by a continuous curve, from which the poles and this portion of the axis are excluded.

It remains to identify this function with the quotient $G(a, \beta, \gamma, x)$ from which Gauss derived the continued fraction. Now

$$G(a, \beta, \gamma, x) = \frac{1}{1+}\,\frac{a_2 x}{1+}\,\frac{a_3 x}{1+}\cdots\cdots\frac{a_{2n} x}{+1 + a_{2n+1}\, x\, G_1},$$

where

$$G_1 = G(a+n, \beta+n, \gamma+2n, x).$$

Hence

$$G(\alpha, \beta, \gamma, x) - \frac{N_{2n}}{D_{2n}} = \frac{N_{2n} + a_{2n+1}\, x\, G_1\, N_{2n-1}}{D_{2n} + a_{2n+1}\, x\, G_1\, D_{2n-1}} - \frac{N_{2n}}{D_{2n}}$$
$$= \frac{a_2\, a_3 \cdot \cdot \cdot a_{2n+1}\, G_1\, x^{2n}}{D_{2n}(D_{2n} + a_{2n+1}\, x\, G_1\, D_{2n-1})}.$$

This shows that when $G(\alpha, \beta, \gamma, x)$ and N_{2n}/D_{2n} are expanded into series in ascending powers of x, their expansions agree to a greater and greater number of terms as n increases. We will prove that $G(\alpha, \beta, \gamma, x)$ coincides with the limit of N_{2n}/D_{2n} by demonstrating the following

Theorem. *If a sequence of analytic functions converges uniformly to a limit in the vicinity of the origin and their expansions in ascending powers of x, as n increases, agree to a greater and greater number of terms with a given series $P(x)$, this series is the limit of the sequence.*[*]

Let $f_n(x)$ represent the nth of these functions and let

$$f_n(x) = a_0^{(n)} + a_1^{(n)}x + \cdot \cdot \cdot \cdot ,$$
$$P(x) = c_0 \;\; + c_1 x + \cdot \cdot \cdot \cdot \cdot .$$

Then, by hypothesis, if the positive integer m is chosen at pleasure, a second integer μ can be so determined that

$$a_0^{(n)} = c_0, \qquad a_1^{(n)} = c_1, \quad \cdot \cdot \cdot \quad a_{m-1}^{(n)} = c_{m-1}, \qquad n \geqq \mu.$$

Hence, generally,

$$\lim_{n - \infty} a_i^{(n)} = c_i.$$

The theorem asserts that

$$P(x) = \lim_{n = \infty} f_n(x).$$

To prove it, then, we have only to show that the limit of the series

$$a_0^{(n)} + a_1^{(n)}x + \cdot \cdot \cdot \cdot$$

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[*] This theorem follows from the theorem that, if $s_n(x)$ is a single valued, analytic function of x throughout the two-dimensional region T of the complex plane for $n = 1, 2, \ldots$ and if $s_n(x)$ converges uniformly in T when $n = \infty$, its limit being denoted by $F(x)$; then $F^{(i)}(x) = \lim_{n=\infty} s_n^{(i)}(x)$, $i = 1, 2, 3, \ldots,$ and the proof is now readily completed. *Cf.* de la Vallée Poussin, *Annales de la Société scientifique de Bruxelles*, seconde partie, Mémoires, vol. 17, 1893, p. 324. [Ed.]

is equal to the series of the limits:

$$\lim_{n=\infty} a_0^{(n)} + \lim_{n=\infty} a_1^{(n)} x + \cdot\cdot\cdot\cdot\cdot .$$

This follows immediately from the uniform convergence. Thus the theorem is proven, and the limit of the continued fraction of Gauss is the function $G(\alpha, \beta, \gamma, x)$.

If α, β, γ are real, an upper limit for the number of poles of this function exterior to the positive half of the real axis can be obtained with the aid of Theorem 3. It will be seen at once that it can not be greater than $2N$, if N is the smallest positive integral value of n for which the following inequalities simultaneously hold:

$$\alpha + n - 1 > 0, \qquad \beta + n > 0$$

$$\gamma - 1 - \beta + n > 0, \qquad \gamma - \alpha + n > 0, \qquad \gamma + 2\,n > 0.$$

This furnishes also an upper limit for the number of zeros of $F(\alpha, \beta, \gamma, x)$ and its analytic continuation which lie without the positive half-axis.

10. (*b*). *The Continued Fraction for Bessel's Quotient* J_n/J_{n-1}. From the well known equation connecting three consecutive related Bessel's functions,

$$\frac{J_{n-1}(x)}{J_n(x)} = \frac{2n}{x} - \frac{J_{n+1}(x)}{J_n(x)},$$

the continued fraction

$$\frac{2n}{x} - \frac{1}{\dfrac{2n+2}{x} -} \; \frac{1}{\dfrac{2n+4}{x} -} \; \cdots$$

is derived. This may be transformed so as to become

$$\frac{2n}{x} - \frac{\dfrac{x}{2n+2}}{1 -} \; \frac{\dfrac{x^2}{(2n+2)(2n+4)}}{1 -} \; \frac{\dfrac{x^2}{(2n+4)(2n+6)}}{1 -} \; \cdots \qquad (18)$$

Also by removing a factor x^{-1} and placing $z = -x^{-2}$, it may be thrown into the form

$$2n + \frac{1}{(2n+2)z +} \; \frac{1}{(2n+4) +} \; \frac{1}{(2n+6)z +} \; \cdots \qquad (19)$$

The application of Theorem 1 to the continued fraction (18) shows that it represents a function which is meromorphic over the entire plane. The identity of the function with J_{n-1}/J_n can be established easily with the aid of the theorem of §9.

When n is real, Theorems 2 and 3 can both be applied to (19). We thus find that the number of poles of the function defined by (19) which lie without the negative half of the axis is equal to zero if $n > -1$. When n lies between the two negative integers, $-m$ and $-(m+1)$, the integer m furnishes an upper limit for the number of such poles. Now to each of these poles there corresponds a pair of imaginary poles of the meromorphic function defined by (18). Since the poles of the latter function are the roots of J_n, we conclude that all the roots of J_n are real if $n > -1$. If $n < -1$, the number of pairs of conjugate imaginary roots can not exceed m. As a matter of fact, this is the exact number.*

11. (*c*). *Heine's Continued Fraction.* One of the generalizations of the hypergeometric series $F(\alpha, \beta, \gamma, x)$, which was proposed by Heine† is the series

$$\phi(\alpha, \beta, \gamma, q, x) = 1 + \frac{(1-q^{\alpha})(1-q^{\beta})}{(1-q)(1-q^{\gamma})} + \frac{(1-q^{\alpha})(1-q^{\alpha+1})(1-q^{\beta})(1-q^{\beta+1})}{(1-q)(1-q^{2})(1-q^{\gamma})(1-q^{\gamma+1})} + \cdots .$$

This satisfies a difference equation of the second order. The hypergeometric series is the limiting case obtained by letting q approach unity. Without in any way restricting the generality of the above series, we may assume q to be less than unity.

For the quotient

$$\frac{\phi(\alpha, \beta+1, \gamma+1, q, x)}{\phi(\alpha, \beta, \gamma, q, x)}$$

Heine obtained a continued fraction of the form (1) in which

$$a_{2n} = q^{\beta+n-1}\frac{(1-q^{\alpha+n-1})(1-q^{\gamma-\beta+n-1})}{(1-q^{\gamma+2n-2})(1-q^{\gamma+2n-1})},$$

$$a_{2n+1} = q^{\alpha+n-1}\frac{(1-q^{\beta+n})(1-q^{\gamma-\alpha+n})}{(1-q^{\gamma+2n-1})(1-q^{\gamma+2n})}.$$

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* *Math. Annalen*, vol. 33, 1888, p. 258.

† *Crelle*, vol. 34, 1847.

When n increases indefinitely, the limit of both of these coefficients is obviously zero. Theorem 1 is therefore applicable and shows that the continued fraction represents a meromorphic function, which, by the theorem of §9, may be proved to be identical with the above quotient.

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